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A SCHUR TYPE INEQUALITY FOR SIX VARIABLES

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ABSTRACT

The purpose of this paper is to deduce a Schur type inequality for six variables.

Keywords: Schur inequality, Schur type inequality for five variables, Schur type inequality for six variables

1 Introduction

Let us consider three real variables $x, y, t \in \mathbb{R}$ such that $x, y \geq 0$ and $t > 0$. Then $x^t(x-y)+y^t(y-x) \geq 0$. Indeed, without loss of generality we can suppose, that $x \geq y \geq 0$. This means $x^t \geq y^t$ for $t > 0$. Consequently $(x-y)(x^t-y^t) \geq 0$ i.e.

$$x^t(x-y)+y^t(y-x) \geq 0. \tag{1}$$

Now let us consider four real variables $x, y, z, t \in \mathbb{R}$ such that $x, y, z \geq 0$ and $t > 0$. Then

$$x^t(x-y)(x-z)+y^t(y-x)(y-z)+z^t(z-x)(z-y) \geq 0, \tag{2}$$

which is known as Schur's inequality, see for example [1]. The idea of Schur in order to prove this inequality is the following: without loss of generality we can suppose $x \geq y \geq z \geq 0$. Then we write this inequality in the form: $(x-y)[x^t(x-z)-y^t(y-z)]+z^t(z-x)(z-y) \geq 0$. Now $x^t \geq y^t \geq 0$ and $x-z \geq y-z \geq 0$ so $x^t(x-z) \geq y^t(y-z)$. This means that $(x-y)[x^t(x-z)-y^t(y-z)] \geq 0$. At the same time $z^t(z-x)(z-y) \geq 0$. By addition we obtain Schur's inequality (2). In [2] we can find different generalizations of the Schur's inequality. We mention that if in (2) we choose $z = 0$ then we reobtain (1) in the form $x^{t+1}(x-y)+y^{t+1}(y-x) \geq 0$.

In [3] we took five real variables $x, y, z, v, t \in \mathbb{R}$ such that $x, y, z, v \geq 0, t > 0$, and we proposed the following Schur's type inequality:

$$\begin{aligned}
&x^t(x-y)(x-z)(x-v)+\\
&+y^t(y-x)(y-z)(y-v)+\\
&+z^t(z-x)(z-y)(z-v)+\\
&+v^t(v-x)(v-y)(v-z) \geq 0.
\end{aligned} \tag{3}$$

Without loss of generality we can suppose that $x \geq y \geq z \geq v \geq 0$. If we choose $x = 6, y = 4, z = 2, v = 1, t = 1$ then (3) is true, but if we choose $x = 10, y = 9, z = 7, v = 2, t = 1$ then (3) is false. In [3] we preserve the Schur type inequality (3) taking a simple supplementary condition:

Theorem 1. *If the real numbers $x, y, z, v, t \in \mathbb{R}$ are such that $x, y, z, v \geq 0, t > 0,$ and $x+v \geq y+z$ then the Schur type inequality (3) is true.*

2 Main part

The purpose of this paper is to generalize Schur's inequality for six variables. Let us consider six real variables $x, y, z, v, w, t \in \mathbb{R}$ such that $x, y, z, v, w \geq 0$ and $t > 0$. Then we propose the following inequality:

$$\begin{aligned}
&x^t(x-y)(x-z)(x-v)(x-w)+\\
&+y^t(y-x)(y-z)(y-v)(y-w)+\\
&+z^t(z-x)(z-y)(z-v)(z-w)+\\
&+v^t(v-x)(v-y)(v-z)(v-w)+\\
&+w^t(w-x)(w-y)(w-z)(w-v) \geq 0.
\end{aligned} \tag{4}$$

Without loss of generality we can suppose that $x \geq y \geq z \geq v \geq w \geq 0$. If we want to use Schur's idea in this case and we try to group the first term with the second or with the third or with the fourth or with the fifth, respectively, then we can not prove this inequality. Indeed, if we denote $F : [0, \infty)^5 \times (0, \infty) \to \mathbb{R}$,

$$\begin{aligned} F(x, y, z, v, w, t) = x^t(x - y)(x - z)\cdot \\ \cdot (x - v)(x - w)+ \\ + y^t(y - x)(y - z)(y - v)(y - w)+ \\ + z^t(z - x)(z - y)(z - v)(z - w)+ \\ + v^t(v - x)(v - y)(v - z)(v - w)+ \\ + w^t(w - x)(w - y)(w - z)(w - v), \end{aligned} \tag{5}$$

then for $n, k \in \mathbb{N}$ sufficiently large natural numbers we have $F(n+4, n+2, n+1, n, 1, k) \geq 0$ and $F(n+4, n+3, n+2, n, 1, k) \leq 0$, because $\lim_{n\to\infty} F(n+4, n+2, n+1, n, 1, k) = +\infty$ for $k \in \mathbb{N}$ sufficiently large natural number and $\lim_{n\to\infty} F(n+4, n+3, n+2, n, 1, k) = -\infty$ for $k \in \mathbb{N}$ sufficiently large natural number. If we use the Newton binomial, then we get $F(n+4, n+2, n+1, n, 1, k) = (n+4)^k \cdot 2\cdot 3\cdot 4\cdot (n+3)+(n+2)^k\cdot(-2)\cdot 1\cdot 2\cdot(n+1)+(n+1)^k\cdot(-3)\cdot(-1)\cdot 1\cdot n+n^k\cdot(-4)\cdot(-2)\cdot(-1)\cdot(n-1)+1^k\cdot(-n-3)\cdot(-n-1)\cdot(-n)\cdot(-n+1) = (24\cdot n^{k+1}+\dots)+(-4\cdot n^{k+1}+\dots)+(3\cdot n^{k+1}+\dots)+(-8\cdot n^{k+1}+\dots)+(n^4+\dots) = (15\cdot n^{k+1}+\dots)+(n^4+\dots)$. If $k \geq 3$ is a fixed natural number, then we have $\lim_{n\to\infty}(15\cdot n^{k+1}+\dots)+(n^4+\dots) = +\infty$, so for $n \in \mathbb{N}$ sufficiently large natural number we have $F(n + 4, n + 2, n + 1, n, 1, k) \geq 0$. But $F(n + 4, n + 3, n + 2, n, 1, k) = (n + 4)^k \cdot 1 \cdot 2 \cdot 4 \cdot (n + 3) + (n + 3)^k \cdot (-1) \cdot 1 \cdot 3 \cdot (n + 2) + (n + 2)^k \cdot (-2) \cdot (-1) \cdot 2 \cdot (n + 1) + n^k \cdot (-4) \cdot (-3) \cdot (-2) \cdot (n - 1) + 1^k \cdot (-n - 3) \cdot (-n - 2) \cdot (-n - 1) \cdot (-n + 1) = (8\cdot n^{k+1}+\dots)+(-3\cdot n^{k+1}+\dots)+(4\cdot n^{k+1}+\dots)+(-24\cdot n^{k+1}+\dots)+(n^4+\dots) = (-15\cdot n^{k+1}+\dots)+(n^4+\dots)$. If $k \geq 3$ is a fixed natural number, then we have $\lim_{n\to\infty}(-15 \cdot n^{k+1}+\dots) + (n^4+\dots) = -\infty$, so for $n \in \mathbb{N}$ sufficiently large natural number we have $F(n + 4, n + 3, n + 2, n, 1, k) \leq 0$.

The aim of this paper is to preserve the Schur type inequality (4) taking a simple supplementary condition:

Theorem 2. *If the real numbers $x, y, z, v, w, t \in \mathbb{R}$ are such that $x \geq y \geq z \geq v \geq w \geq 0, t > 0$ and $x + v \geq y + z$, then the Schur type inequality (4) is true.*

Proof. Let us denote $a = x - y \geq 0, b = y - z \geq 0, c = z - v \geq 0, d = v - w \geq 0$. The condition $x + v \geq y + z$, means $x - y \geq z - v$, i.e. $a \geq c$. Now we rewrite the left side of (4) using (5) and the variables a, b, c, d :

$$\begin{aligned} F(x, y, z, v, w, t) = x^t \cdot a(a + b)(a + b + c)\cdot \\ \cdot (a + b + c + d)+ \\ + y^t \cdot (-a)b(b + c)(b + c + d)+ \\ + z^t \cdot [-(a + b)](-b)c(c + d)+ \\ + v^t \cdot [-(a + b + c)][-(b + c)](-c)d+ \\ + w^t \cdot [-(a + b + c + d)][-(a + b + c)][-(c + d)](-d) = \end{aligned}$$

$$\begin{aligned} = (a + b + c) \cdot [x^t \cdot a(a + b)(a + b + c + d)- \\ - v^t \cdot (b + c)cd]- \\ - b \cdot [y^t \cdot a(b + c)(b + c + d)- \\ - z^t \cdot (a + b)c(c + d)]+ \\ + w^t \cdot (a + b + c + d)(a + b + c)(c + d)d. \end{aligned} \tag{6}$$

We will demonstrate that

$$\begin{aligned} x^t \cdot a(a + b)(a + b + c + d) - v^t \cdot (b + c)cd \geq \\ \geq y^t \cdot a(b + c)(b + c + d) - z^t \cdot (a + b)c(c + d) \\ \Leftrightarrow x^t \cdot a(a + b)(a + b + c + d)+ \\ + z^t \cdot (a + b)c(c + d) \\ \geq y^t \cdot a(b + c)(b + c + d) + v^t \cdot (b + c)cd. \end{aligned} \tag{7}$$

But using the conditions of theorem 2 we get $x^t \geq y^t \geq 0, a \geq 0, a + b \geq b + c \geq 0, a + b + c + d \geq b+c+d \geq 0$, so $x^t \cdot a(a+b)(a+b+c+d) \geq y^t \cdot a(b+c)(b + c + d)$ and $z^t \geq v^t \geq 0, a + b \geq b + c \geq 0, c \geq 0, c+d \geq d \geq 0$, so $z^t \cdot (a+b)c(c+d) \geq v^t \cdot (b+c)cd$. Adding term by term these two inequalities we receive (7).

Next we prove

$$\begin{aligned} y^t \cdot a(b + c)(b + c + d) - z^t \cdot (a + b)c(c + d) \geq 0 \\ \Leftrightarrow y^t \cdot a(b + c)(b + c + d) \geq \\ \geq z^t \cdot (a + b)c(c + d) \geq 0. \end{aligned} \tag{8}$$

Indeed, $y^t \geq z^t \geq 0, a \geq c \Rightarrow ab \geq bc \Rightarrow ab + ac \geq ac + bc \Rightarrow a(b + c) \geq (a + b)c \geq 0, b + c + d \geq c + d$, which implies the above inequality.

Now we summarize and from (7) and (8) we can deduce that $x^t \cdot a(a+b)(a+b+c+d) - v^t \cdot (b+c)cd \geq y^t \cdot a(b + c)(b + c + d) - z^t \cdot (a + b)c(c + d) \geq 0$. At the same time $a + b + c \geq b \geq 0$ so $(a + b + c) \cdot [x^t \cdot a(a + b)(a + b + c + d) - v^t \cdot (b + c)cd] \geq b \cdot [y^t \cdot a(b + c)(b + c + d) - z^t \cdot (a + b)c(c + d)]$.

To finish the proof we observe that $w^t \cdot (a + b + c + d)(a + b + c)(c + d)d \geq 0$, so from (6) we have $F(x, y, z, v, w, t) \geq 0$. $\square$

Remark 1. *Using the hypothesis of theorem 2 we can observe that in the Schur type inequality (4) we realize the equality in the following cases:*

i) $x = y = z = v = w \geq 0$;
ii) $x = y = z = v \geq 0$ *and* $w = 0$;
iii) $x = y = z \geq 0$ *and* $v = w = 0$;
iv) $x = y \geq 0$ *and* $z = v = w = 0$.

Remark 2. *If we choose $w = 0$ then using theorem 2 and condition $x + v \geq y + z$ from the Schur type inequality (4) we reobtain theorem 1 and the inequality (3) in the form*

$$\begin{aligned} x^{t+1}(x - y)(x - z)(x - v)+ \\ + y^{t+1}(y - x)(y - z)(y - v)+ \\ + z^{t+1}(z - x)(z - y)(z - v)+ \\ + v^{t+1}(v - x)(v - y)(v - z) \geq 0. \end{aligned} \tag{9}$$

Now if we choose $v = 0$ using theorem 1 with condition $x \geq y + z$ from the above inequality we get

$$x^{t+2}(x-y)(x-z) + y^{t+2}(y-x)(y-z) + \\ + z^{t+2}(z-x)(z-y) \geq 0, \tag{10}$$

which is similar to the Schur's inequality (2).

3 Discussion and conclusion

We can generalize the results of theorem 1 and theorem 2 using [2]. We can obtain a new Schur type inequality for seven variables, if we impose two simple supplementary conditions.

References

[1] http://en.wikipedia.org/wiki/Schur%27s_inequality

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